Part I Calculus（60%）

1. Evaluate the iterated integral $\int_0^1\left[\int_y^1 e^{x^2}dx\right]dy$. (10%)

2. (a) Let f be a function from an interval $I \subseteq \mathbb{R}$ onto an interval $J \subseteq \mathbb{R}$. Suppose that f has an inverse f^{-1} such that $f'(x)$ exists for $x \in I$, and $\left(f^{-1}\right)'(x)$ exists for $x \in J$. Show that

$$\left(f^{-1}\right)'(x) = \frac{1}{f'(f^{-1}(x))},$$

for each $x \in J$ for which $f'(f^{-1}(x)) \neq 0$ holds. (5%)

(b) It is known that $(\sec x)' = \sec x \tan x$. Prove that

$$\left(\sec^{-1} x\right)' = \frac{1}{|x|\sqrt{x^2-1}} \quad \text{for } |x|>1.$$

Note that $\sec^{-1} x \in \left[0, \pi/2\right) \cup \left(\pi/2, \pi\right]$. (10%)

3. (1) Let $f:[a,b] \to [a,b]$ be a continuous function. Show that

(a) $|f(b)-f(a)| \le |b-a|$. (5%)

(b) f has a fixed point x in [a, b]; that is, there exists $x \in [a,b]$ such that f(x) = x. (10%)

(2) Let f: $\mathbb{R} \to \mathbb{R}$ be a contraction; that is, $|f(x)-f(y)| \le \alpha|x-y|$ for all $x, y \in \mathbb{R}$ and for some constant α with $0<\alpha<1$. Show that

(a) f is uniformly continuous on $\mathbb{R}$. (5%)

(b) Let $x_0 \in \mathbb{R}$ be fixed. Define $x_n = f(x_{n-1})$, $n = 1, 2, \cdots$. Then $\{x_n\}$ converges to a point c in $\mathbb{R}$. (10%)

(c) f has a fixed point in $\mathbb{R}$; that is, there exists $c \in \mathbb{R}$ such that $f(c) = c$. (5%)

Part II Linear algebra (40%)

4. Let A be the 3×3 matrix

$$\begin{bmatrix} 1 & 1 & 0 \\ -1 & 3 & 0 \\ -1 & 1 & 2 \end{bmatrix}.$$

Find the eigenvalues and eigenspaces of A. Show that A is not diagonalizable but there is an upper triangular matrix that is similar to A. (20%)

5. Suppose A is a 4×4 matrix such that the solution space of $A^T x = 0$ is spanned by the column vector $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$. Here A^T is the transpose of A. Find a basis for the orthogonal complement of the column space of A and deduce that the solution space of $Ax = \begin{bmatrix} 1 \\ 2 \\ 2 \\ 4 \end{bmatrix}$ is the empty set. (20%)