國立彰化師範大學 99 學年度博士班招生考試試題

系所：數學系　　　　選考甲　　　　科目：高等微積分

☆☆請在答案紙上作答☆☆　　　　共 1 頁，第 1 頁

1.(15%) Prove or disprove the convergence of following integrals:

(a) $0<p<1, \int_2^\infty x^{-1}(\ln x)^{-p}\,dx$　　(b) $\int_0^\infty e^{-x}\sin\left(x^{-1}\right)dx$　　(c) $p>0, \int_1^\infty x^{-p}\cos x\,dx$

2. (a) (9%) Find the Taylor expansion at $x=0$ for the function $f(x)=\sin^{-1}x$. What is its radius of convergence? Hint: consider $f'(x)$.

(b) (6%) Show $\pi=3\left(1+\sum_{k=1}^{\infty}\frac{1\cdot\cdot(2k-1)}{(2k+1)2^{3k}k!}\right)$

3. (15%) Suppose $A=\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{bmatrix}$ is symmetric positive definite. Show that

$$\iint_{\mathbf{R}^2} e^{-\left(a_{11}x^2+a_{12}xy+a_{21}xy+a_{22}y^2\right)}dxdy=\frac{\pi}{\sqrt{a_{11}a_{22}-a_{12}a_{21}}}$$

4. (20%) Suppose $f(x,t)$ is continuous for all (x,t) with $x\in E$ and $t\ge c$, where E is an open subset of $\mathbf{R}^n$. Let $\int_c^\infty f(x,t)\,dt$ converges to $F(x)$ uniformly for all x in E. Show F is continuous on E.

5. Let $\mathbf{X}$ be a compact metric space and let f be a map of $\mathbf{X}$ onto itself. Suppose that there exist $0<\alpha<1$ such that $d\left(f(x),f(y)\right)\le\alpha\cdot d(x,y)$. Prove that

(a) (8%) $(\mathbf{X},d)$ is a complete metric space.

(b) (9%) f has a unique fixed point $\bar{x}$, that is, $f(\bar{x})=\bar{x}$.

6. Let $\{x_n, n=1,2...\}$ be a bounded sequence in $\mathbf{R}^n$. Let $g:\mathbf{R}^n\to\mathbf{R}$ be a function defined by

$$g(z)=\overline{\lim_{n\to\infty}}\|x_n-z\|,\ \text{for all } z\in\mathbf{R}^n$$

(a) (9%) Show that g is continuous.

(b) (9%) if $u,v\in\mathbf{R}^n$, $0\le\lambda\le1$, show that $g\left(\lambda u+(1-\lambda)v\right)\le\lambda g(u)+(1-\lambda)g(v)$.