國立彰化師範大學 98 學年度博士班招生考試試題

系所：數學系　　　　　　　　　　　　　　　　　　　　科目：高等微積分

☆☆請在答案紙上作答☆☆　　　　　　　　　　　　　　　　

(1) Let $f: R \to R$ be a continuous function on R, show that the set $A = \{(x,y) \mid f(x) > y\}$ is open in R^2. (10%)

(2) If $a_n > 0$ for all $n \in N$, show that $\limsup_{n\to\infty} \sqrt[n]{a_n} \le \limsup_{n\to\infty} \frac{a_{n+1}}{a_n}$.(10%)

(3) Show that $\lim_{n\to\infty}(\int_0^{\pi} \sin^n x \, dx)^{1/n} = 1$. (10%)

(4) Let $f_n(x) = x^n$ and $g(x) = \cos(\frac{\pi}{2}x)$ for $x \in R$.

(a) Show that the sequence $\{f_n\}$ converges pointwise but not uniformly on [0,1]. (10%)

(b) Show that the sequence $\{g(x)x^n\}$ converges uniformly on [0,1]. (10%)

(5) Assume that $\sum_{n=1}^{\infty} a_n$ converges absolutely, show that $\sum_{n=1}^{\infty} \frac{a_n{}^3}{1+a_n{}^2}$ converges absolutely. (10%)

(6) Let $f: R \to R$ be a continuous function on R.

(a) If $f([a,b]) \supset [a,b]$, show that f has a fixed point in $[a,b]$. (10%)

(b) If $f(f(f(0))) = 0$, show that f has a fixed point in R. (10%)

(7) A function $f:[a,b] \to R$ is said to satisfy a uniform Lipschitz condition of order α on [a,b] , if there exists a constant C>0 such that $|f(x) - f(y)| < C\,|x-y|^{\alpha}$ for all x and y in [a,b].

(a) Let f satisfy a uniform Lipschitz condition of order 1 on [a,b], show that f is of bounded variation on [a,b]. (10%)

(b) Please find a function f satisfying a uniform Lipschitz condition of order $\frac{1}{2}$ on [a,b] such that f is not of bounded variation on [a,b]. (10%)