國立彰化師範大學105學年度博士班招生考試試題

系所： 數學系　　組別： 甲組選考乙　　科目： 代數

☆☆請在答案紙上作答☆☆　　

1. (16%) Let $\mathbb{R}$ be the field of real numbers and $F = \left\{ \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \middle|\ a, b \in \mathbb{R} \right\}$.

(1) Prove that F is an abelian group under the usual matrix addition.
(2) Prove that F is a ring under the usual matrix addition and multiplication.
(3) Is F a field? Justify your answer.

2. (12%) Suppose that G is a finite group of order p, where p is a prime number. Prove that G is a cyclic group.

3. (12%) Suppose that G is an abelian group and n is a fixed positive integer. Let $A_n = \{a^n \mid a \in G\}$. Prove that A_n is a subgroup of G.

4. (12%) Let R be a ring such that $a^2 = a$ for all $a \in R$. Prove that R is a commutative ring and $a + a = 0$ for all $a \in R$.

5. (12%) Let G be a group and $a, g \in G$. Prove that $|g^{-1}ag| = |a|$. Here $|x|$ is the order of x in G.

6. (12%) Prove that if $N \triangleleft G$ (i.e. N is a normal subgroup of G) and H is any subgroup of G then $N \cap H \triangleleft H$.

7. (12%) Denote the center of a group G by $Z(G)$. Suppose $G/Z(G)$ is cyclic. Show that G is abelian.

8. (12%) Let H be a subgroup of G. Suppose the index $[G:H]$ of H in G is 2, then H is a normal subgroup of G. Deduce then A_n is a normal subgroup of S_n.