國立彰化師範大學103學年度博士班招生考試試題

系所：數學系　　組別：甲組（選考乙）　　科目：代數

☆☆請在答案紙上作答☆☆　　

1. Let $K \subseteq H \subseteq G$ be groups. If $|H:K| = n$ and $Kh_1, Kh_2, \ldots, Kh_n$ are the distinct cosets of K in H. Show that $Hg = Kh_1g \cup Kh_2g \cup \cdots \cup Kh_ng$ is a disjoint union for all $g \in G$. (16%)

2. Let H and K be normal subgroups of a group G.
Define $HK = \{hk \mid h \in H, k \in K\}$. Show that $HK \lhd G$. (16%)

3. Prove that $\mathbb{R}/\mathbb{Z} \cong \mathbb{C}^0$, where $\mathbb{C}^0 = \{z \in \mathbb{C} : |z| = 1\}$. (18%)

4. Define a new addition and multiplication on $\mathbb{Q}$ by

$$r \oplus s = r + s - 1,$$
$$r \odot s = rs - r - s + 2.$$

Prove that $(\mathbb{Q}, \oplus, \odot)$ is a commutative ring with unity. Is it a field? (16%)

5. Suppose R is a commutative ring with unity and $a \in R$.
Prove that $Ra = \{ra \mid a \in R\}$ is an (two-sided) ideal of R containing a. (16%)

6. Determine all ideals of $\mathbb{Z}$ and all quotient (factor) rings of $\mathbb{Z}$. (18%)