國立彰化師範大學 102 學年度博士班招生考試試題

系所：數學系　　選考乙　　科目：代數

☆☆請在答案紙上作答☆☆　　

1. Let $G=\langle a\rangle$ be the cyclic group of order n generated by a. Show that the order of the element $a^s\in G$ is $n/\gcd(s,n)$. (10%)

2. Let $D_4=\{1,a,a^2,a^3,b,ba,ba^2,ba^3\}$ be the octic group, where $|a|=4$, $|b|=2$, and $aba=b$. Find the center Z of D_4 and determine D_4/Z. (20%)

3. Let $G=\left\{\begin{pmatrix} a & b \\ 0 & c\end{pmatrix} \middle|\ a,b,c\in\mathbb{R}; a\neq 0, c\neq 0\right\}$. If $K=\left\{\begin{pmatrix} 1 & b \\ 0 & 1\end{pmatrix} \middle|\ b\in\mathbb{R}\right\}$, show that $K\lhd G$ and $G/K\cong\mathbb{R}^*\times\mathbb{R}^*$. Here $\mathbb{R}^*=\mathbb{R}-\{0\}$. (20%)

4. (a) Let L and M be intermediate fields of the extension $K\subset F$, of finite dimensional over K. Suppose $[LM:K]=[L:K][M:K]$. Prove that $L\cap M=K$.

(b) The converse of (a) holds if $[L:K]$ or $[M:K]$ is 2.

(total 20%)

5. Let I be an ideal of the commutative ring R. Define

$$\operatorname{rad} I=\{r\in R\mid r^n\in I \text{ for some positive integer } n\}. \quad (15\%)$$

(a) Prove that $\operatorname{rad} I$ is an ideal containing I.

(b) Suppose I is a prime ideal of R. Show that $\operatorname{rad} I=I$.

6. Let I be an ideal of $R\times S$ direct product of rings R and S. Let

$$I_R=\{a\in R\mid \text{there is } b\in S \text{ so that } (a,b)\in I\},$$
$$I_S=\{b\in S\mid \text{there is } a\in R \text{ so that } (a,b)\in I\}.$$

Prove that I_R, I_S are ideals of R and S, respectively; and $I=I_R\times I_S$. (15%)