國立彰化師範大學 98 學年度博士班招生考試試題

系所：數學系　　　　　　　　　　　　　科目：線性代數

☆☆請在答案紙上作答☆☆　　　　　　　　　共 1 頁，第 1 頁

1. Let $A=\begin{pmatrix}2 & 6\\ 6 & -3\end{pmatrix}$. Find an orthogonal matrix C such that $C^{-1}AC$ is diagonal. (20%)

2. Let A be an $n\times n$ matrix over a field F, that is, $A\in M_n(F)$. Prove that A is invertible if and only if $AB=I$ for some $B\in M_n(F)$, where I is the identity matrix. (15%)

3. If $T:V\to W$ is a linear mapping and N is the kernel of T. Prove that
   (a) $T^{-1}(\{Tx\})=x+N$, for all $x\in V$; (10%)
   (b) $V/N\cong T(V)$, via the mapping $x+N\to Tx$. (10%)

4. Let A be a complex $n\times n$ matrix such that $A^3-2A^2-A-6I=0$, where I is the identity matrix.
   (a) Determine whether A is invertible or not. Explain the reason. (10%)
   (b) Determine whether A is diagonalizable or not. Explain the reason. (10%)

5. (a) Let A and B be two complex $n\times n$ matrices. Show that λ is an eigenvalue of AB if and only if λ is an eigenvalue of BA. (10%)
   (b) Let A and B be two complex $n\times n$ matrices such that $AB=BA$. Show that A and B have at least one eigenvector in common. That is, there is a $n\times 1$ vector $\vec{v}$ such that $\vec{v}$ is not only an eigenvector of A but also an eigenvector of B. (10%)

6. Let V be a finite-dimensional complex vector space and $T:V\to V$ a linear transformation. Suppose that W is a subspace of V. Show that there is a nonzero complex polynomial $f(x)=\sum_{i=0}^{k}a_ix^i$ such that $f(T)W\subseteq W$, where $f(T)=a_kT^k+a_{k-1}T^{k-1}+\cdots+a_1T+a_0I$ and I is the identity transformation on V. (5%)