國立彰化師範大學 96 學年度博士班招生考試試題

系所：數學系　　　　　　　　　　　　　　　　科目：線性代數

☆☆請在答案紙上作答☆☆　　　　　　　　　　　　共 1 頁，第 1 頁

1. Let $M_{3\times3}(R)$ be the set of all 3×3 real matrices.

   Let $T: M_{3\times3}(R) \to M_{3\times3}(R)$ be the linear transformation defined by $T(A) = A^t$ for all

   $A \in M_{3\times3}(R)$, where A^t is the transpose of A.

   (a) Show that ±1 are the only eigenvalues of A. (5%)
   (b) Find all eigenspaces of A. (10%)
   (c) Is T diagonalizable ? Explain the reason. (5%)
   (d) Is T invertible ? Find T^{-1} if your answer is positive. (5%)

2. Let V be a finite-dimensional vector space over a field F and $T: V \to V$ a linear transformation. Let $R(T)$ denote the range space of T and $N(T)$ the null space of T. Prove that the dimension of $N(T)$ plus the dimension of $R(T)$ equals the dimension of V. That is, $\dim N(T) + \dim R(T) = \dim V$. (15%)

3. Let A and B be $n\times n$ real matrices. Show that if A is invertible, then there exists infinitely many real number α such that $A+\alpha B$ is also invertible. (12%)

4. Let $T: V \to V$ be a linear transformation on a vector space V over a field F. Suppose that the dimension of V is n and T has n distinct eigenvalues in F.
   (a) Prove that the minimal polynomial of T is also the characteristic polynomial of T. (12%)
   (b) Show that there exists a $\vec{v} \in V$ such that $\vec{v}, T(\vec{v}), T^2(\vec{v}), ..., T^{n-1}(\vec{v})$ are linear independent over F. (15%)

5. Let V be a n-dimensional vector space over a field F.

   (a) Show that V is isomorphic to the n-tuples space $F^n = \{(a_1, a_2, ..., a_n) \mid a_1, a_2, ..., a_n \in F\}$. (6%)

   (b) Show that if F is a finite field with q elements, then the number of different bases of V over F is $(q^n-1)(q^n-q)(q^n-q^2)\cdots(q^n-q^{n-1})$. (15%)