國立彰化師範大學九十六學年度碩士班招生考試試題

系所：資訊工程學系　　　　　　　　　　　　　　　　科目：離散數學及線性代數

☆☆請在答案紙上作答☆☆　　　　　　　　　　　　　　　　

1. Let the *height* of a full binary tree T denoted as $h(T)$ is defined recursively as follows:
   BASIC STEP: The height of the full binary tree consisting of only a root r is $h(T)=0$.
   RECURSIVE STEP: If T_1 and T_2 are full binary trees, then the full binary tree $T = T_1 \cdot T_2$ has height $h(T)=1+\max(h(T_1), h(T_2))$.

   Moreover, let $n(T)$ denoted the number vertices in a full binary tree and $n(T)$ is defined recursively as follows:
   BASIC STEP: The number vertices $n(T)$ of the full binary tree T consisting of only a root r is $n(T)=1$.
   RECURSIVE STEP: If T_1 and T_2 are full binary trees, then the number vertices $n(T)$ of the full binary tree $T = T_1 \cdot T_2$ is $n(T) = 1 + n(T_1) + n(T_2)$.
   Note: $T = T_1 \cdot T_2$ means that the binary tree T consists of a root r together with edges connecting the root to each of the roots of the left subtree T_1 and the right subtree T_2.

Prove that if T is a full binary tree, then $n(T) \leqq 2^{h(T)+1} - 1$. (10%)

2. Find the solution to the recurrence relation $a_n = 6a_{n-1} - 11a_{n-2} + 6a_{n-3}$ with $a_0 = 2$, $a_1 = 5$ and $a_2 = 15$? (10%)

3. Prove that if G is a *connected planar simple graph* with e edges and v vertices, where $v \geqq 3$, then $e \leqq 3v-6$. (10%)

4. Compute 3^{644} $\boldsymbol{mod}$ 645. (10%)

5. Prove that for any integer $n>0$, there exists a sequence of n consecutive *composite* integers. (10%)

6. Let A be a 4×4 matrix defined as follows

$$A = \begin{bmatrix} 1/2 & -1/2 & -1/2 & -1/2 \\ -1/2 & 1/2 & -1/2 & -1/2 \\ -1/2 & -1/2 & 1/2 & -1/2 \\ -1/2 & -1/2 & -1/2 & 1/2 \end{bmatrix}$$

   Find A^n. (10%)

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7. Let $\boldsymbol{A}$ be a 4×4 matrix. If $\boldsymbol{adjA}$ (*adjoint* of matrix $\boldsymbol{A}$) is known as follows, find $\boldsymbol{A}$. (10%)

$$\text{adj A} = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 4 & 3 & 2 \\ 0 & -2 & -1 & 2 \end{bmatrix}$$

8. Let $\{\mathbf{u_1}, \mathbf{u_2}, \mathbf{u_3}\}$ be an *orthornormal basis* for an inner product space $\boldsymbol{V}$ and let $\boldsymbol{u} = \boldsymbol{u_1} + \boldsymbol{2u_2} + \boldsymbol{2u_3}$ and $\boldsymbol{v} = \boldsymbol{u_1} + \boldsymbol{7u_3}$. Compute (1) $<\boldsymbol{u}, \boldsymbol{v}>$ and (2) the angle $\boldsymbol{\theta}$ between $\mathbf{u}$ and $\mathbf{v}$. (10%)

9. Prove that (1) the sum of the *eigenvalues* of an $n\times n$ matrix $\boldsymbol{A}$ is equal to the sum of the diagonal elements of $\boldsymbol{A}$, (2) the product of the *eigenvalues* of an $n\times n$ matrix $\boldsymbol{A}$ is equal to $\boldsymbol{det(A)}$. (10%)

10. Compute $\boldsymbol{e^A}$ for the matrix $\boldsymbol{A}$. (10%)

$$A = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 3 \\ 1 & 1 & -1 \end{bmatrix}$$