1. (15%) Calculate the circulation of vector function $\mathbf{F} = (x-y)\mathbf{i} + x^2 y\mathbf{j} + xza\mathbf{k}$ counterclockwise about the circle $x^2 + y^2 = 1$. Here a is a positive constant.

2. (15%) Use the Fourier Transform to solve $y'' + 6y' + 5y = \delta(t-3)$.

3. (20%) Legendre's equation is $(1-t^2)y'' - 2ty' + \mu(\mu+1)y = 0$. This equation has a power series solution of the form $y(t) = \sum_{n=0}^{\infty} a_n t^n$ that is guaranteed to be absolutely convergent in the interval $-1 < t < 1$.

   (a) Find the recurrence relation for the coefficients of the power series.

   (b) Argue, when $\mu = N$ is a nonnegative integer, that Legendre's equation has a polynomial solution, $P_N(t)$.

   (c) Use the recurrence relation and the requirement that $P_N(1) = 1$ to determine the first three Legendre polynomials, $P_1(t)$, $P_2(t)$, $P_3(t)$.

4. (20%) Consider the Strum-Liouville boundary value problem $y'' + 2y' + (1+\lambda)y = 0$, $y(0) = y(1) = 0$.

   (a) Classify the Strum-Liouville boundary value problem as regular, periodic or singular; state the relevant interval.

   (b) Find the eigenvalues λ_n and eigenfunctions $\phi_n(x)$.

   (c) Find the eigenfunction expansion of the function $f(x) = \begin{cases} 0, & 0 \le x \le 0.5 \\ 1, & 0.5 < x \le 1 \end{cases}$ in the eigenfunctions of the Strum-Liouville problem.

   (d) Determine what the eigenfunction expansion converges to on the interval.

5. (15%) Consider the initial value problem $y'' + by' + cy = f(x)$, $0 < t < \infty$, $y(0) = y_a$ and

$y'(0) = y_b$. The input function $f(t) = e^{-t}$ and the Laplace transform of the output function

$y(t)$ is $Y(s) = \dfrac{s^2 + s + 1}{(s+1)(s^2+4)}$. Determine the constants of b, c, y_a, y_b.

6. (15%) Find the Cauchy principal value of $\displaystyle\int_{-\infty}^{+\infty} \frac{\cos(3x)}{x-1}\,dx$.