1. Roll a fair die repeatedly. Let T denote the number of rollings before all 6 sides appeared at least once. Compute $E(T)$ and $Var(T)$. (20%)

2. The joint probability density function of X and Y is given by

$$f(x, y) = \begin{cases} 6xy(x + y) & 0 \le x \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

   Find $P(X + Y \le 1)$ and $E(X + Y)$. (20%)

3. Let U be a random variable with uniform distribution on $(0,1)$. (20%)

   (a) Find a function $g(x)$ defined on $(0,1)$ such that $g(U)$ has a Poisson distribution with parameter 1.

   (b) Find a function $h(x)$ defined on $(0,1)$ such that $h(U)$ has an exponential distribution with parameter 1.

4. Consider a random sample of size n from a uniform distribution, $X_i \sim U(-\theta, \theta)$, $\theta > 0$, and let $X_{1:n}$ and $X_{n:n}$ be the smallest and the largest order statistics, respectively. (20%)

   (a) Find the probability that the random interval $(X_{n:n}, 2X_{n:n})$ contains θ.

   (b) Let $Y_n = \max\{-X_{1:n}, X_{n:n}\}$. Find the constant c such that (Y_n, cY_n) is a level $1 - \alpha$ confidence interval for θ.

5. Consider a random sample of size n from a normal distribution $N(0, \sigma^2)$. (20%)

   (a) Derive the critical region for a uniformly most powerful test of size α of $H_0 : \sigma = \sigma_0$ against $H_a : \sigma > \sigma_0$.

   (b) Find the power function for the test in (a).