國立彰化師範大學 102 學年度碩士班招生考試試題

系所：數學系　　　　　　　　　　　　　　　　科目：線性代數

☆☆ 請在答案紙上作答☆☆　　　　　　　　　　　共 1 頁，第 1 頁

1. Let

$$V=\{(x_1,x_2,x_3,x_4,x_5)\in \mathbf{R}^5 : x_1-2x_2+x_3-x_4+2x_5=0\}.$$

Show that $S=\{(-2,0,1,-1,0),(7,1,-3,0,-1)\}$ is a linearly independent subset of V and extend S to be a basis for V. (20%)

2. Let V and W be vector spaces and $T:V\to W$ be linear. Suppose that $\beta=\{v_1,v_2,\ldots,v_n\}$ is a basis for V and T is one-to-one and onto. Prove that $T(\beta)=\{T(v_1),T(v_2),\ldots,T(v_n)\}$ is a basis for W. (15%)

3. Let A and B be $n\times n$ matrices with real entries such that AB is invertible. Prove that A and B are invertible. (15%)

4. Find the determinant of the $n\times n$ matrix. (15%)

$$A=\begin{bmatrix} 0 & -1 & 0 & \cdots & 0 \\ 1 & 0 & -1 & \cdots & 0 \\ 0 & 1 & 0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & -1 \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}.$$

5. Determine whether the matrix

$$A=\begin{bmatrix} 3 & 1 & 1 \\ 2 & 4 & 2 \\ -1 & -1 & 1 \end{bmatrix}$$

is diagonalizable. If A is diagonalizable, find an invertible matrix Q that diagonalizes A. (15%)

6. Let $A\in M_{n\times n}(\mathbb{C})$ be a self-ajoint matrix, that is, $A^*=A$. The standard inner product on $\mathbb{C}^n$ is given by $\langle \mathbf{u},\mathbf{v}\rangle=\mathbf{u}^*\mathbf{v}$. (20%)

(a) Prove $\langle A\mathbf{u},\mathbf{v}\rangle=\langle \mathbf{u},A\mathbf{v}\rangle$.

(b) Use (a) or otherwise, show that eigenvalues of A are real.

(c) Let $\mathbf{u}$ be a λ-eigenvector of A and $\mathbf{v}$ be a μ-eigenvector of A. Prove that if $\lambda\neq\overline{\mu}$, then $\langle \mathbf{u},\mathbf{v}\rangle=0$.