

# 國立彰化師範大學 97 學年度碩士班招生考試試題

系所：數學系碩士班

組別：甲/乙/丙/丁組

科目：線性代數

☆☆請在答案紙上作答☆☆

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請回答下列題目

1. Let  $U$ ,  $V$ , and  $W$  be subspaces of some vector space  $X$  and  $U$  is a subset of  $W$ .

(a) Show that  $U+V=\{u+v \mid u \in U, v \in V\}$  is a subspace of  $X$ . (10 %)

(b) Show that  $(U+V) \cap W=U+(V \cap W)$ . (10 %)

2. Let  $T: \mathbb{R}^n \rightarrow \mathbb{R}^n$  be a linear transformation satisfying  $T(T(x))=2T(x)+8x$ , for all  $x$  in  $\mathbb{R}^n$ .

Is  $T$  one-to-one? Is  $T$  onto? Explain. (15 %)

3. For any matrix, show that elementary row operations do not change the dimension of the column space. (15 %)

4. A sequence of vectors  $\{v_n\}_{n=0}^{\infty}$  in  $\mathbb{C}^2$  ( $\mathbb{C}$  is the field of complex numbers) is defined as follows:

(15%)

$$v_{n+1} = Av_n, \quad n = 0, 1, 2, \dots, \text{ where } A = \begin{bmatrix} 1 & -1+i \\ 1 & 0 \end{bmatrix}.$$

(a) Determine  $A^n$ .

(b) Find  $v_{100}$  if  $v_0 = \begin{bmatrix} 1 \\ i \end{bmatrix}$ .

5. Let  $V$  be a subspace of  $\mathbb{R}^n$ . A linear transformation  $T: V \rightarrow V$  is said to be symmetric if  $(Tu, v) = (u, Tv)$  for all  $u, v \in V$ . Here  $(x, y)$  is the dot product of vectors  $x$  and  $y$ . A subspace  $W$  of  $\mathbb{R}^n$  is said to be invariant under  $T$  if  $Tw \in W$ , for all  $w \in W$ .

(a) Show that  $T$  is symmetric if and only if the matrix representation of  $T$  relative to some orthonormal basis is symmetric. (10%)

(b) Show that if  $W$  is invariant under  $T$ , then the orthogonal complement  $W^\perp$  of  $W$  is also invariant under  $T$ . (10%)

6. (15%)

(a) Show that similar matrices have the same characteristic polynomial, hence they have the same set of eigenvalues. Discuss how their eigenspaces are related.

(b) Find two matrices which have the same set of eigenvalues but they are not similar (explanation is required).