國立彰化師範大學 102 學年度碩士班招生考試試題

系所：數學系　　組別：乙組　　科目：高等微積分

☆☆請在答案紙上作答☆☆

1. Let $f: R \to R$ be a continuous periodic function on R with period 2π. Prove that there exists an $x_0 \in R$ such that $f(x_0) = f(x_0 + \pi)$. (10%)

2. (1) Let $f_n(x) = \dfrac{1-e^{-nx}}{\sqrt{x}}$ *on* $(0, \infty)$. Determine $f(x) = \lim\limits_{n\to\infty} f_n(x)$ *for* $0 < x < \infty$, and Show that f_n converges uniformly to f on $[a, \infty)$ $(0 < a < \infty)$. (15%)

   (2) Let $g_n(x) = nxe^{-nx}$ *on* $[0, \infty)$. Determine $g(x) = \lim\limits_{n\to\infty} g_n(x)$ *for* $0 \le x < \infty$, and prove that g_n does not converge uniformly to g *on* $[0, \infty)$. (15%)

3. (1) Show that $f(x) = \dfrac{8}{\pi}\sum\limits_{k=1}^{\infty} \dfrac{\sin(2k-1)x}{(2k-1)^3}$ is continuous on R. (10%)

   (2) Show that $f(x)$ is differentiable on R. (10%)

4. Evaluate the following limits :

   (1) $\lim\limits_{n\to\infty} \displaystyle\int_0^{2\pi} \frac{\cos(nx^2)}{x^2+n^2}\,dx$. (10%)

   (2) $\lim\limits_{h\to 0} \dfrac{1}{h} \displaystyle\int_3^{3+h} e^{2x^2}\,dx$. (10%)

5. Let $f(x, y) = \begin{cases} \dfrac{x^3+y^3}{x^2+y^2} & \text{for } (x, y) \ne (0, 0) \\ 0 & \text{for } (x, y) = (0, 0) \end{cases}$ (20%)

   Find $\dfrac{\partial f}{\partial x}(0,0)$, $\dfrac{\partial f}{\partial y}(0,0)$ and the directional derivative of f at (0, 0) in the direction of unit vector $u = (u_1, u_2)$. Is $f(x, y)$ continuous at $(0, 0)$? Prove your answer.